

# First Results from Lattice Simulation of the PWMM

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## Abstract

We present results of lattice simulations of the Plane Wave Matrix Model (PWMM). The PWMM is a theory of supersymmetric quantum mechanics that has a well-defined canonical ensemble. We simulate this theory by applying rational hybrid Monte Carlo techniques to a naïve lattice action. We examine the strong coupling behaviour of the model focussing on the deconfinement transition.

# 1 Introduction

The AdS/CFT correspondence [1, 2, 3] has emerged as a useful setting for studying certain strongly coupled gauge theories. Though this correspondence has been widely tested, it is useful to look for examples in which the correspondence can be tested directly by studying theories in which tractable calculations can be done directly at strong coupling. Lower dimensional examples would seem to provide a useful setting for such a study. Recent effort has been devoted to studying the BFSS matrix model at strong coupling using both lattice and non-lattice techniques [4, 5, 6]. However, even that one dimensional quantum theory is still not well defined in the canonical ensemble because it has a moduli space [7]. The moduli space in this theory is given by the eigenvalues of nine commuting matrices that transform under the  $SO(9)$  R-symmetry. The existence of these flat directions means that, at finite temperature, the partition function is formally divergent, and it was shown in [7] that when Monte Carlo simulations of this theory are performed, this divergence eventually causes the simulation to break down.

The Plane Wave Matrix Model (PWMM) [8] arises as a natural next choice for simulation because it includes supersymmetry preserving mass terms that lift the moduli space, and give a discrete set of vacua. The theory still has enough supersymmetry that it has a known gravity dual at zero temperature [9]. Symmetry reduces the dual supergravity problem to a two dimensional one, and the geometries dual to the discrete set of field theory vacua can be put in one-to-one correspondence with a set of axisymmetric electrostatics configurations, the potential of which determines the supergravity fields, and was solved in [10]. Moreover, this theory has a number of large  $N$  limits that give theories of physical interest: M-theory on a plane-wave [8], little string theory on  $S^5$  [11], and  $\mathcal{N} = 4$  SYM on  $R \times S^3$  [12]. Studying the PWMM at strong coupling, therefore, is a first step toward direct simulation of these theories.

The PWMM is also a good candidate for establishing benchmarks for studies of lattice supersymmetry in general. It is simple enough to allow for direct study using a naïve lattice action, is well-defined in the canonical ensemble and of course being one dimensional is numerically very tractable. As such it is complementary to studies of supersymmetric theories in higher dimensions using new lattice formulations which retain some exact supersymmetry at non zero lattice spacing - see the recent review [13]. For example it should be possible to cross check the results of direct simulations of  $\mathcal{N} = 4$  super Yang-Mills in certain limits with results derived from simulations of naïve lattice discretizations of the PWMM.

In this paper we will present the results of simulations of the PWMM on the lattice. We use a naïve lattice action, and concentrate on studying the Hagedorn/deconfinement transition in the model. We study the theory at fixed temperature, measured in units of the mass deformation, as a function of the 't Hooft coupling (measured in the same units). We used a quenched approximation to explore the dependence of the critical behaviour on the rank of the gauge group and the number of lattice points. The quenched approximation is much less computationally demanding and can be used to estimate reasonable choices of parameters for simulating the full theory. We establish that we

can get a reasonable approximation of the continuum large  $N$  behaviour with modest values of both  $N$  and the size of the lattice. We then simulate the full theory. The Hagedorn/deconfinement transition in the PWMM has also been studied previously at weak coupling 't Hooft coupling in the large  $N$  limit, and we compare our results to extrapolations of those from weak coupling.

The remainder of this paper is organized as follows. In section 2 we review the PWMM, its vacuum solutions, and various limits. In section 3 we present our lattice action and discuss the simulation method. Section 4 contains our main results. We review the weak coupling results and contrast them with the results of our simulations. In section 5 we discuss possible completions of the phase diagram.

## 2 The PWMM

The Plane-Wave Matrix Model (PWMM) is a gauged matrix quantum mechanics theory with sixteen supercharges. It has an action that can be written as

$$S = \frac{N}{2\lambda} \int_0^\beta dt \text{Tr} \left( - \sum_i (DX_i)^2 - \sum_{i < j} [X^i, X^j]^2 + \Psi^T \gamma^0 D\Psi + \sum_i \Psi^T \gamma_i [X^i, \Psi] \right. \\ \left. - \mu^2 \sum_{i=1}^3 (X^i)^2 - \frac{\mu^2}{4} \sum_{i=4}^9 (X^i)^2 - 2\sqrt{2}\mu \epsilon_{ijk} X^i X^j X^k + \frac{3\mu}{4} \Psi^T \gamma_{123} \Psi \right), \quad (1)$$

in which  $X^i$  are nine scalar matrices that sit in the adjoint representation of  $SU(N)$ , i.e.  $X^i = \sum_{a=1}^{N^2-1} X_i^a T^a$ , where  $T^a$  are anti-Hermitian generators that are normalized to  $\text{Tr } T^a T^b = -\delta_{ab}$ ;  $\Psi$  is a Majorana fermion that is also in the adjoint representation of the gauge group;  $\text{Tr}$  is over the gauge indices;  $D$  is a gauge covariant derivative;  $\lambda$  is the 't Hooft coupling, given by  $\lambda = g^2 N$ ; and

$$\gamma_{123} = \frac{1}{6} \epsilon_{ijk} \gamma_i \gamma_j \gamma_k. \quad (2)$$

The terms in the first line of (1) are those in the BFSS matrix model [14]. The terms in the second line give supersymmetry preserving masses to the fields and add a Myers term. The addition of these terms gives this model two technical advantages compared with BFSS. Firstly, in the limit  $\mu \rightarrow \infty$  the model becomes weakly coupled and can be studied perturbatively [15]. Also, the addition of these terms lifts the moduli space of the BFSS model giving instead a discrete set of vacua [8]. This is important for numerical simulation because it means the theory is well-defined in the canonical ensemble. If we consider the scalar fields  $X^i$  for  $i = 1, 2, 3$  we can write the potential as

$$V = - \text{Tr} \left[ (\mu X^i + \frac{1}{\sqrt{2}} \epsilon_{ijk} [X^j, X^k])^2 \right], \quad (3)$$

which is minimized when the scalar fields  $X^i$  are proportional to generators of  $SU(2)$ . The discrete vacua correspond to the various ways of forming  $N \times N$  matrices that

generate  $SU(2)$ , and can be put in one-to-one correspondence with partitions of  $N$  into a sum of natural numbers. It was shown in [16] that the vacua sit in certain doubly atypical representations of the  $SU(2|4)$  supersymmetry algebra, which protect them from quantum corrections, and thus persist at strong coupling.

The identification between the field theory vacuum states and regular type IIA supergravity solutions with  $SU(2|4)$  supersymmetry was found in [9].  $SU(2|4)$  contains a bosonic  $R \times SO(3) \times SO(6)$  that reduces the type IIA supergravity problem to a two-dimensional one. Imposing supersymmetry and regularity reduces the two-dimensional problem to solving a two-dimensional Laplace equation. The vacua of the PWMM are encoded by choosing appropriate boundary conditions. Using the fact that the potential in electrostatics obeys the Laplace equation, it was shown [9] that regular supergravity solutions correspond to axisymmetric configurations of charged conducting discs. This class of electrostatics problems was solved in [10].

The PWMM has a number of large  $N$  limits in which it describes interesting physics. See [11] for a summary; we will mention a few here here.<sup>1</sup>

1. If we consider a limit in which

$$N \rightarrow \infty \quad \frac{g^2}{N^3} \text{ fixed}, \quad (4)$$

this model should describe M-theory in a maximally supersymmetric plane-wave background. This is because the matrix theory conjecture suggests this model describes the DLCQ of M-theory on the plane wave with compact null momentum given by

$$\mu l_p^2 p^+ = \frac{N}{g^{\frac{2}{3}}} \propto \frac{N}{R}, \quad (5)$$

where  $R$  is the size of the null circle. The limit (4) decompactifies the compact null circle.

2. The 't Hooft limit, in which  $N \rightarrow \infty$  with  $\lambda$  fixed, allows for the perturbative study of the theory if  $\lambda$  is small. It is in this limit that the Hagedorn/deconfinement transition can be studied analytically, as we will review below.
3. It was shown in [11] that if we consider expanding around the trivial vacuum solution and take a limit in which, asymptotically,

$$N \rightarrow \infty \quad \frac{\ln^4 N}{\lambda} \text{ fixed}, \quad (6)$$

the PWMM describes type IIA little string theory on  $S^5$ . Little string theory is the theory that describes the degrees of freedom on NS5-branes, and is of particular interest because it exhibits some of the features of string theories (e.g. T-duality), but not others (e.g. gravity).

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<sup>1</sup>For links between the PWMM and other field theories see [17, 18].

4. Using arguments from the gravity side [9] or from the field theory side [12]  $\mathcal{N} = 4$  SYM on  $R \times S^3$  can be realized by expanding about a particular vacuum of the PWMM. (For a full discussion, see also [19], [20].) Expanding about a vacuum consisting of representations of  $SU(2)$  of dimension  $n+j$ , for some integers  $n$ , and  $j = 1, 2, 3, \dots, m$ , each with  $k$  copies. If the PWMM has gauge group  $SU(N)$ , then we must have

$$N = \frac{1}{2}km(2n + m + 1). \quad (7)$$

Taking a limit  $N \rightarrow \infty$  in which

$$n \rightarrow \infty \quad m \rightarrow \infty \quad \frac{m}{n} \rightarrow 0, \quad (8)$$

gives  $\mathcal{N} = 4$  SYM on  $R \times S^3$  with gauge group  $SU(k)$ .

It is in principle possible to extract the physics of these models by studying the behaviour of the model along appropriate curves in the  $\lambda - N$  plane, and extrapolating to large  $N$ . In this paper we seek to establish benchmarks for this procedure by performing a lattice simulation of the PWMM at strong coupling and finite  $N$ .

### 3 Lattice Action

To find our lattice action, we start with the PWMM action (1). The bare action has three parameters with unit mass dimension  $\lambda^{\frac{1}{3}}$ ,  $\beta^{-1}$  and  $\mu$ . The gauge field and scalars  $X^i$  also have unit mass dimension, and the fermions have mass dimension  $\frac{3}{2}$ . We would like to make the following rescaling:

$$X \rightarrow \mu T X \quad D \rightarrow \mu T D \quad t \rightarrow \frac{t}{\mu T} \quad \Psi \rightarrow (\mu T)^{\frac{3}{2}} \Psi, \quad (9)$$

where  $T$  is a dimensionless number that we will shortly take to be the number of lattice points. We can then write the action as

$$\begin{aligned} S = \frac{\mu^3 T^3 N}{2\lambda} \int_0^{\mu \beta T} dt \text{Tr} \Big( & - \sum_i (DX_i)^2 - \sum_{i < j} [X^i, X^j]^2 + \Psi^T \gamma^0 D \Psi + \sum_i \Psi^T \gamma_i [X^i, \Psi] \\ & - \frac{1}{T^2} \sum_{i=1}^3 (X^i)^2 - \frac{1}{4T^2} \sum_{i=4}^9 (X^i)^2 - 2\sqrt{2} \epsilon_{ijk} X^i X^j X^k + \frac{3}{4T} \Psi^T \gamma_{123} \Psi \Big). \end{aligned} \quad (10)$$

Note that all of our fields have been rendered dimensionless. After integrating over the fermions we have the following partition function

$$Z = \int dA dX \text{Pf}(\mathcal{O}) e^{-S_B}, \quad (11)$$

with

$$S_B = \frac{\mu^3 T^3 N}{2\lambda} \int_0^{\mu\beta T} dt \text{Tr} \left( - \sum_i (DX_i)^2 - \sum_{i < j} [X^i, X^j]^2 - \frac{1}{T^2} \sum_{i=1}^3 (X^i)^2 - \frac{1}{4T^2} \sum_{i=4}^9 (X^i)^2 - 2\sqrt{2}\epsilon_{ijk} X^i X^j X^k \right), \quad (12)$$

and

$$\mathcal{O} = \gamma^0 D + \sum_i \gamma_i [X^i, \cdot] + \frac{3}{4T} \gamma_{123}. \quad (13)$$

We discretize the theory on a lattice with spacing  $a$ , and replace the  $\int dt$  with  $a \sum_t$ . We use the derivative operator

$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}, \quad (14)$$

where

$$D^+ \Psi = \frac{1}{a} (U(t) \Psi(t+a) U^\dagger - \Psi(t)). \quad (15)$$

$D^-$  is the adjoint of  $D^+$ . The lattice action is

$$S_B = \frac{\beta \mu^4 T^3 N}{2\lambda} \sum_{m=0}^{T-1} \text{Tr} \left( - \sum_i (DX_m^i)^2 - \sum_{i < j} [X_m^i, X_m^j]^2 - \frac{1}{T^2} \sum_{i=1}^3 (X_m^i)^2 - \frac{1}{4T^2} \sum_{i=4}^9 (X_m^i)^2 - 2\sqrt{2}\epsilon_{ijk} X_m^i X_m^j X_m^k \right), \quad (16)$$

and

$$\mathcal{O} = \gamma^0 D_{mn} + \sum_i \gamma_i [X_m^i, \cdot] \mathbb{1}_{mn} + \frac{3}{4T} \gamma_{123} \mathbb{1}_{mn}. \quad (17)$$

The overall dimensionless lattice coupling is then

$$\kappa = \frac{\beta \mu^4 T^3 N}{2\lambda}, \quad (18)$$

where  $\beta$  is the inverse temperature, and  $T$  is the number of lattice points. The continuum limit is defined by taking  $T \rightarrow \infty$  with  $\frac{\mu^4 N \beta}{2\lambda}$  fixed.

We simulate this model using Rational Hybrid Monte Carlo (RHMC) [21]. Since the fermions appeared quadratically, we were able to formally integrate them out yielding the Pfaffian of the operator  $\mathcal{O}$ . Using the fact that  $\text{Pf}^2(\mathcal{O}) = \det \mathcal{O}$ , the Pfaffian can be computed by introducing a path integral over pseudofermions  $F$

$$\text{Pf } \mathcal{O} = \int [DF] e^{-F^\dagger (\mathcal{O}^\dagger \mathcal{O})^{-\frac{1}{4}} F}. \quad (19)$$

It is still challenging to compute  $(\mathcal{O}^\dagger \mathcal{O})^{-\frac{1}{4}}$ , but it can be done efficiently by making a rational approximation, and making use of iterative Krylov subspace methods to solve the resulting set of shifted linear systems [22]. Fictitious conjugate momenta are introduced for each of the variables, and the resulting system is simulated by using a combination of molecular dynamics integrations of the equations of motion and Metropolis Monte Carlo [23, 24]. The main computational expense derives from simulating the fermionic Pfaffian, and so it will be useful at times to consider the quenched approximation (without fermions). This will allow us to estimate what happens at much larger values of  $N$  than we can when the fermions are included with equivalent computational effort.

## 4 Hagedorn/Deconfinement Transition

To get an idea of how well our finite  $N$  simulations work we will compute a simple observable in the matrix model: the Polyakov loop. There are good reasons to compute it.

- It is a simple gauge invariant observable that is straightforward to define on the lattice.
- It is an order parameter for deconfinement, and is expected to signify a geometric transition in the dual gravity theory.
- It has been computed in the PWMM at weak coupling.

In lattice gauge theory, holonomies of the gauge field are natural observables. In our one-dimensional theory, the holonomy to compute is the one around the thermal circle. It can serve as an order parameter for deconfinement because it measures the energy cost of adding an infinitely massive external charged particle. In the confined phase, the trace of the Polyakov loop has vanishing expectation value. This is because it is a unitary matrix, and in the confined phase its eigenvalues are uniformly distributed on the unit circle. In the deconfined phase this symmetry is broken, the eigenvalues begin to clump and the Polyakov loop develops a vev.

It is also interesting from the point of view of gauge/gravity duality. In gravity, there is a well-known phase transition between thermal AdS space and asymptotically AdS-Schwarzschild black holes [25]. Black hole thermodynamics implies that this phase transition comes with an increase in entropy. It was argued [26] that this fact, along with the fact that the Wilson line computed via gauge/gravity duality ceases to obey an area law in the high temperature phase, means this bulk gravity transition corresponds to deconfinement in the gauge theory. The increase in gravitational entropy also matches nicely with this picture because in the confined phase the free energy is expected to be of order one, whereas in the deconfined phase it is expected to be of order  $N^2$ . For a discussion of general aspects of this, see [27]. We therefore expect that by studying the deconfinement transition at strong coupling in the matrix model we are capturing a geometric transition in the bulk dual.

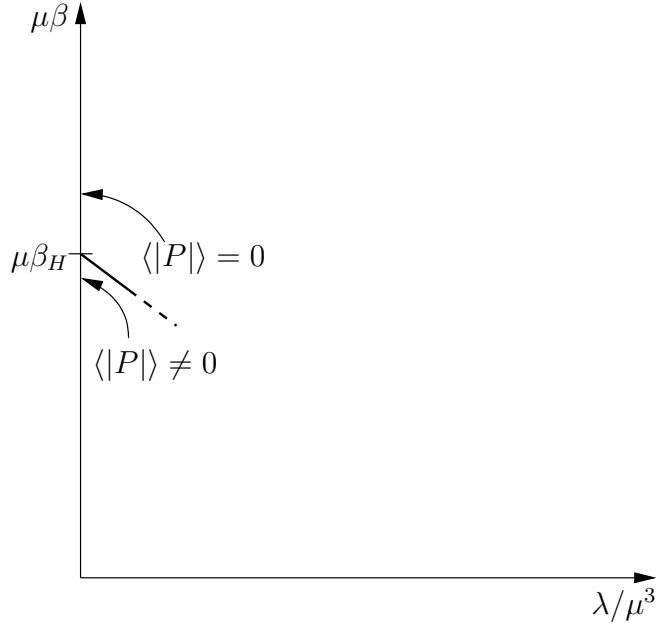


Figure 1: Schematic picture of the phase diagram of the PWMM at weak coupling. For  $\mu\beta$  above  $\mu\beta_H$  (at low temperature) the theory is confined. As  $\mu\beta$  is lowered through  $\mu\beta_H$  the theory undergoes a Hagedorn/deconfinement transition. The transition temperature can be computed at weak coupling; the calculation in the quenched theory is reviewed in section 4.1.1, and section 4.2.1 presents the result in the full theory.

In [27, 28] it was shown that in certain large  $N$  gauge theories, including the PWMM, it is possible to have deconfinement transitions at weak coupling. Let us discuss briefly why this occurs, using an argument from [27]. Consider a theory of two bosonic matrix oscillators with unit energy in the adjoint representation of a gauge group with rank  $N$ . Gauge invariant states are formed by the action of traces of matrix creation operators on the vacuum. Let us consider single trace states, which dominate in the large  $N$  limit. The number of single trace states with energy  $E$  is bounded above by  $2^E$ , because there are  $E$  positions in the trace that can be filled by either oscillator. Some of these states are related by the cyclicity of the trace, but there are at most  $E - 1$  such states, and so the number of states with energy  $E$  is bounded below by  $\frac{2^E}{E}$ . This lower bound is sufficient to produce an exponentially growing density of states, which indicates that there will be a Hagedorn transition.

The existence of results at weak coupling suggests where we should look for the phase transition at strong coupling. A schematic picture of the phase diagram at weak coupling is shown in figure 1. We will see below that the weak coupling analysis suggests that  $\mu\beta_H$  decreases as the coupling is turned on. As a result, for our simulations, we will pick some fixed  $\mu\beta < \mu\beta_H$  and look for the transition by picking various values of the coupling constant. We will first examine what happens in the quenched theory, and thereafter present results for the full theory.

## 4.1 Quenched Theory

Here we will first present results for the quenched theory at weak coupling, which can be culled from [28, 29], and then the results of our simulations.

### 4.1.1 Weak Coupling

By counting gauge invariant states, it was argued in [28] that a matrix model with action

$$S = \int dt \sum_{j=1}^d \frac{1}{2} \text{Tr} \left[ \left( \frac{d}{dt} X_j + i[A, X_j] \right)^2 - \omega_j^2 X_j^2 \right], \quad (20)$$

with bosonic matrices  $X_j$  in the adjoint of  $SU(N)$ , and  $A$  being an  $SU(N)$  gauge field, has a partition function that, in the large  $N$  limit, is approximately

$$Z(\beta) \approx \frac{e}{1 - \sum_{j=1}^d e^{-\beta\omega_j}}. \quad (21)$$

This partition function clearly diverges when the denominator vanishes, and it was argued in [28] that the temperature where this happens is the Hagedorn temperature.

$$\sum_{j=1}^d e^{-\beta_H \omega_j} = 1. \quad (22)$$

If we consider the various fields in the PWMM and decompose them into representations of  $SU(2|4)$  which contains a bosonic  $SU(2) \times SU(4)$ , there are scalars that transform as a singlet under  $SU(4)$  and in the 3 of  $SU(2)$  and scalars that transform as a singlet under  $SU(2)$  and in the 6 of  $SU(4)$ . If we expand the quenched theory about the trivial vacuum state, we then have three matrices with  $\omega = \mu$  and six matrices with  $\omega = \mu/2$ . Writing  $x_H = e^{-\beta_H \mu/2}$  we find the Hagedorn temperature is given by the solution of

$$3x_H^2 + 6x_H = 1, \quad (23)$$

where the admissible solution satisfies  $0 \leq x_H \leq 1$ . This gives  $x_H = -1 + \frac{2}{\sqrt{3}}$ , so that

$$\mu\beta_H = -2 \log \left( -1 + \frac{2}{\sqrt{3}} \right) \approx 3.732528074. \quad (24)$$

Since the only massless field in (20) is the holonomy of the gauge field around the thermal circle, it is possible to integrate out the matrices to find an effective action. This effective action is just the effective action for the Polyakov loop. At one-loop level it was shown in [28] that there is a deconfinement transition, whose temperature coincides with the Hagedorn temperature.

It is also possible to calculate the effective action of the Polyakov loop to higher order in the coupling constant. This has been carried out in the full theory to two-loop order in [29], and three-loop order in [30]. The two-loop calculation is sufficient to

determine the correction to the critical temperature, whereas three-loops are needed to determine the order of the phase transition.

The next order correction to the Hagedorn temperature comes from the planar two-loop effective action for the Polyakov loop. In this case, using the results of [29], the partition function gets corrected to<sup>2</sup>

$$Z = \frac{e}{1 - 3y^4 - 6y^2 - \tilde{\lambda}g(y) \log y}, \quad (25)$$

with

$$g(y) = 48y^2(y^6 + 3y^4 + 7y^2 + 13), \quad (26)$$

where  $y = e^{-\beta\mu/4}$ , and we write a dimensionless 't Hooft coupling  $\tilde{\lambda} = \lambda/\mu^3$ . Let us define

$$z(y) = 3y^4 + 6y^2. \quad (27)$$

The correction to the transition temperature is given by

$$y_H = y_H^{(0)}(1 - c \ln y_H^{(0)} \tilde{\lambda}), \quad (28)$$

or, equivalently,

$$\mu\beta_H = -4 \log(y_H^{(0)}) (1 - c\tilde{\lambda}), \quad (29)$$

for a constant  $c$  that is determined by

$$c = \frac{g(y_H^{(0)})}{y_H^{(0)} z'(y_H^{(0)})}. \quad (30)$$

From (29) it can be seen that  $c > 0$  decreases the inverse temperature as the coupling is turned on, whereas  $c < 0$  increases it. Evaluating this at  $y_H^{(0)} = \sqrt{x_H}$  gives

$$c \approx 49.04614622, \quad (31)$$

which means that the inverse temperature at the transition decreases as the coupling is turned on.

#### 4.1.2 Strong Coupling

Here we will study the quenched theory by expressing all length scales in terms of  $\mu$ . The quenched theory is useful for exploring some of the key issues of our lattice methods. The PWMM describes interesting physics in various limits in which the rank of the gauge group,  $N$ , is taken to infinity. Since the phase space of the model grows very quickly with  $N$  it is necessary to understand if interesting physics is within the range of  $N$  that is computationally accessible. Moreover, we are also limited by the lattice approximation itself and we would like to determine how many lattice points are required to obtain a reasonable approximation to the continuum.

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<sup>2</sup>There is also a correction of order  $\tilde{\lambda}$  coming from the non-planar diagrams, but it does not contribute to the correction to the Hagedorn temperature, so we don't include it.

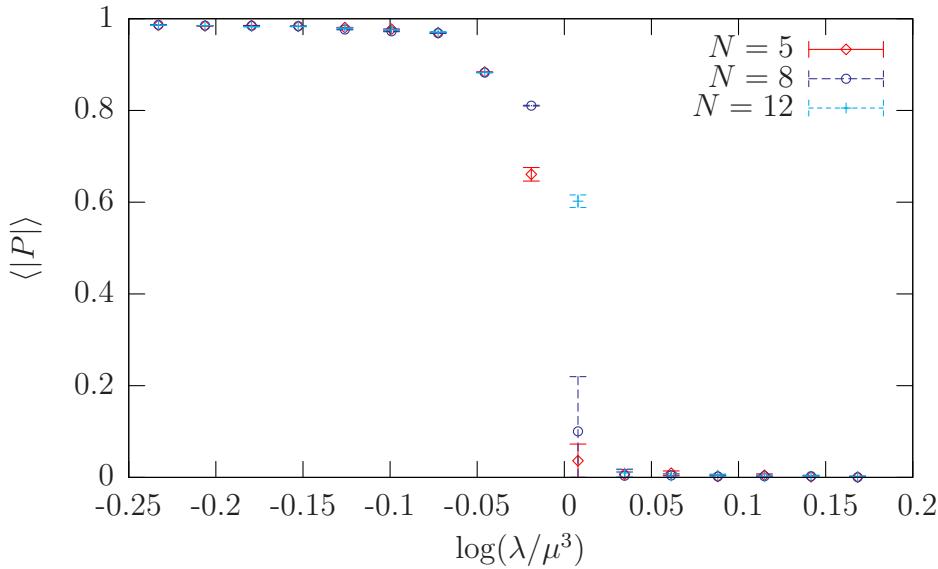


Figure 2:  $\langle |\text{Tr } e^{i \oint A}| \rangle$  in the quenched theory for gauge group  $SU(N)$  with various  $N$ , and  $T = 5$  lattice points. The results agree quite well for all choices of  $N$  above the transition, and well below the transition. The greatest difference is near the transition, where it appears to be sharper in the case of larger  $N$ , as expected.

Figure 2 shows a plot of the Polyakov loop in the quenched theory as a function of the lattice coupling measured in units of  $\mu$ , with the inverse temperature  $\beta\mu = 1$ , for various  $N$  with five lattice points. The plot shows clear evidence of a deconfinement transition. Moreover, it can be seen from this plot that the critical value of the coupling does not seem to be very dependent on  $N$ . This is very suggestive that the critical coupling in the large  $N$  theory can be reasonably approximated by simulations with modest values of  $N$ . Note, however, the transition does appear to be much sharper for larger  $N$ , as expected.

In figure 3 we plot the Polyakov line in the quenched theory as a function of the coupling at fixed  $\beta\mu = 1$  and  $N = 5$  for various lattice sizes,  $T$ . The plot indicates that the critical value of the coupling does not depend strongly on the lattice size, and that  $T = 5$  provides a reasonably good approximation to the continuum.

Let us compare these results with those at weak coupling. If we extrapolate the weak coupling results to  $\mu\beta = 1$ , we can use (29) to estimate the critical value of the coupling. We find  $\log(\lambda/\mu^3) \approx -1.826043339$ . Comparing this value to the plot in figure 2, we see that the critical value of the coupling differs considerably.

There are various possible reasons for this difference. One is that the weak coupling computation was done in the 't Hooft limit expanding about the trivial vacuum state. The results of our simulations, in contrast, should include the effects of all of the various vacua, and are for finite  $N$ . Thinking about our results from the weak coupling point of view, as we change  $N$  our results could be sensitive to both the number of oscillators above each vacuum state, and the number of vacuum states, which change with  $N$ .

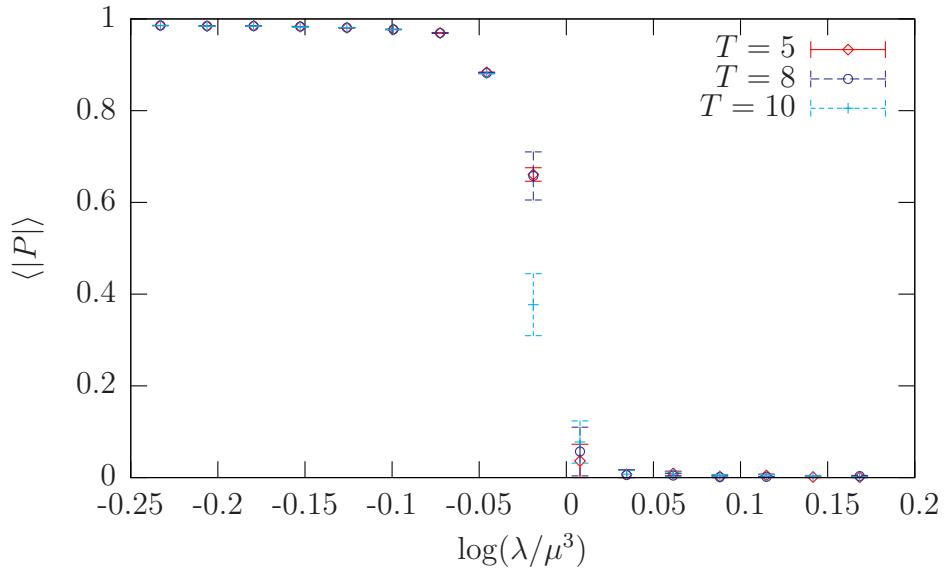


Figure 3:  $\langle |\text{Tr } e^{i \oint A}| \rangle$  in the quenched theory for various  $T$ ,  $N = 5$ .

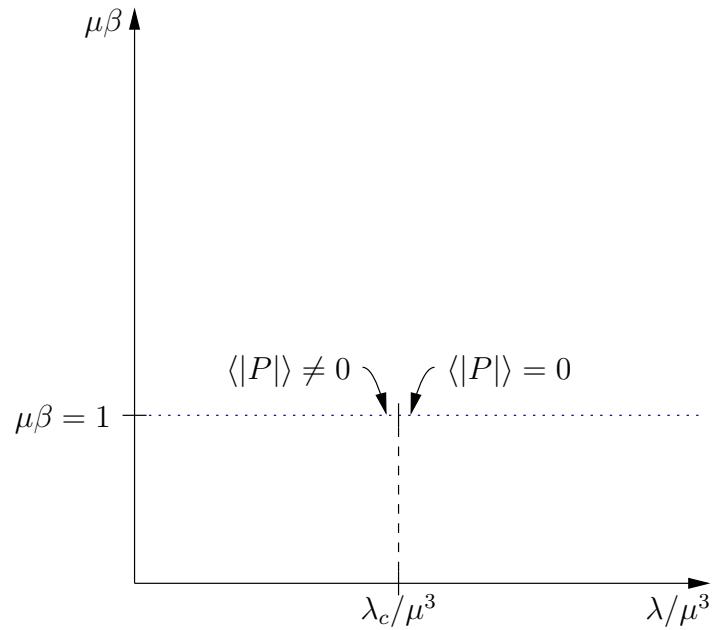


Figure 4: Schematic picture of the phase diagram from our lattice results. We study the theory along the blue dotted line, and identify a critical coupling,  $\lambda_c$ . At strong coupling (at this temperature) the theory is confined. It undergoes a deconfinement transition at  $\lambda_c$ . Compare this with the phase diagram at weak coupling in figure 1.

We would expect that both of these effects would show up in the  $N$  dependence of our results, yet figure 2 shows the only prominent  $N$  dependence is in the sharpness of the transition.

## 4.2 Full Theory

### 4.2.1 Weak Coupling

The thermodynamics of multi-matrix models with fermions was also considered in [28]. It was shown that for free gauged matrix quantum mechanics with a collection of bosonic and fermionic matrices with frequencies  $\omega_j$  the partition function in the large  $N$  limit is approximately

$$Z(\beta) \approx \frac{e}{1 - \sum_{j=1}^d e^{-\beta\omega_j}}. \quad (32)$$

I.e. (21) continues to hold, however the sum now runs over the frequencies of both the bosonic and fermionic matrices. Again, the inverse Hagedorn temperature is given by the place where the partition function diverges. Moreover, as in the quenched case this can be shown to coincide with the deconfinement transition temperature computed from the one loop effective action for the Polyakov loop.

If we again consider classifying the fields by their transformations under  $SU(2|4)$ , in addition to the bosonic scalars we had in the quenched case we now also have fermions that transform in the fundamental representation of both  $SU(2)$  and  $SU(4)$ . If we expand about the trivial vacuum state, then, we have, in addition to the three bosonic matrices with  $\omega = \mu$  and six with  $\omega = \mu/2$ , eight fermionic matrices with  $\omega = 3\mu/4$ . Writing  $y = e^{-\beta\mu/4}$  we find the Hagedorn temperature is given by the solution of

$$3y_H^4 + 8y_H^3 + 6y_H^2 - 1 = (3y_H - 1)(y_H + 1)^3 = 0, \quad (33)$$

where  $y = y_H$  at the transition and the admissible solution satisfies  $0 \leq y_H \leq 1$ . This gives  $y_H = \frac{1}{3}$ , so that

$$\mu\beta_H = 4 \log 3 \approx 4.394449156. \quad (34)$$

Comparison with the result in the quenched theory (24) gives a difference of about 15%.

As in the quenched case, the next order correction to the Hagedorn temperature comes from the planar two-loop effective action for the Polyakov loop. Here, again using results of [29], the partition function gets corrected to

$$Z = \frac{e}{1 - z(y) - \tilde{\lambda}g(y)\log y}, \quad (35)$$

with

$$z(y) = 3y^4 + 8y^3 + 6y^2 \quad g(y) = 48y^2(1+y)^4(1+y^2). \quad (36)$$

As in the quenched case, the correction to the Hagedorn temperature, written in the form (29) gives (30) with

$$c = \frac{205}{81}. \quad (37)$$

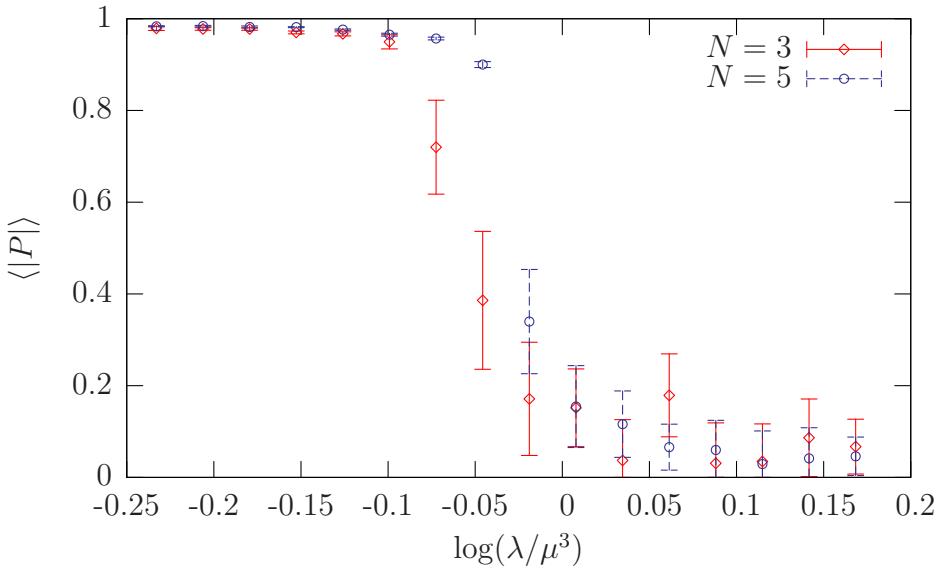


Figure 5:  $\langle |\text{Tr } e^{i \oint A}| \rangle$  In the full theory with dynamical fermions. The gauge group is  $SU(N)$  for  $N = 3, 5$ , and we are using five lattice points.

Comparing this to the result in the quenched theory, (31), shows that the correction to the inverse temperature of the transition is stronger by about an order of magnitude in the quenched case compared to the full theory.

#### 4.2.2 Strong Coupling

We study the PWMM at strong coupling using lattice techniques, as in the quenched case. However, the inclusion of the fermionic Pfaffian substantially increases the computational expense. Fortunately, the results of the quenched simulations suggest that five lattice points provide a reasonable approximation to the continuum, at the temperature we are considering.

Let us compare these results with those at weak coupling. If we extrapolate the weak coupling results to  $\mu\beta = 1$ , we can use (29) to estimate the critical value of the coupling. We find  $\log(\lambda/\mu^3) \approx -0.5154039782$ . Comparing this value to the plot in figure 5, we see that the actual critical value of the coupling differs considerably. Another interesting order parameter for the transition is shown in figure 6 which plots the expectation value of the energy in dimensionless units as a function of the dimensionless 't Hooft coupling. This is essentially given by the value of the bosonic action since the contributions of the fermions can be computed exactly by a scaling argument. Clearly, for strong coupling or small mass the vacuum is supersymmetric  $\langle \frac{E}{\mu} \rangle = 0$  while supersymmetry apparently spontaneously breaks for small 't Hooft coupling.

If we compare the critical coupling of the full theory as plotted in figure 5 with that of the quenched theory plotted in figure 2 we see that there is little difference. This might be surprising, considering that extrapolation of the weak coupling results suggested they should differ by more than an order of magnitude. We will comment

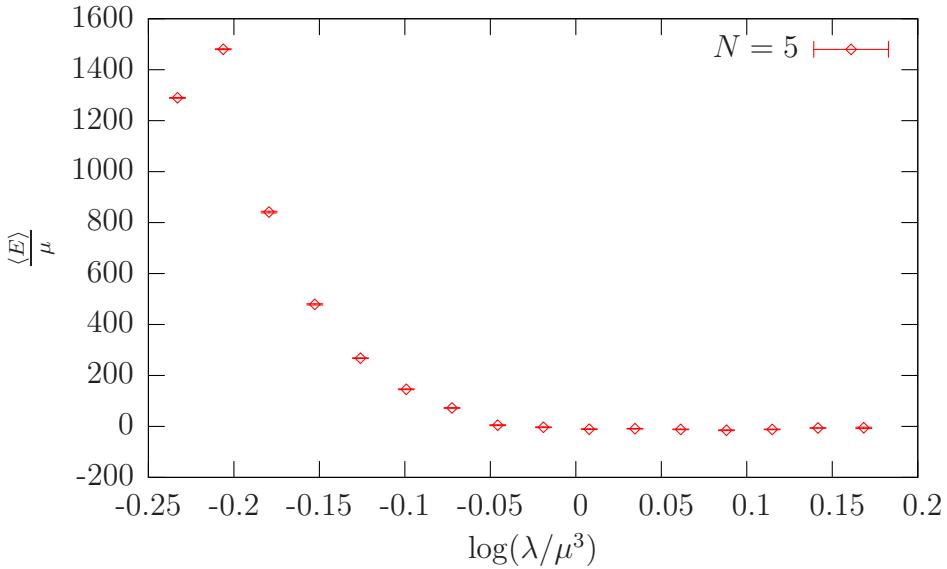


Figure 6: Energy  $\frac{E}{\mu}$  vs  $\log \lambda/\mu^3$  for  $SU(5)$

on this in the next section.

A potential pitfall of our numerical simulations is the notorious sign problem. In figure 7 we plot the Pfaffian phase for the theory with gauge group  $SU(3)$ . The expectation value of the phase is very close to zero, as expected, but more importantly, the uncertainty in the phase is very small.

## 5 Discussion

We have presented results on lattice simulations of the PWMM. We argued that the mass terms that deform the BFSS matrix model to the PWMM render it particularly suitable for lattice study by lifting the moduli space and yielding a discrete set of vacua. We have focussed on studying the Hagedorn/deconfinement transition in the model. By simulating the model at fixed temperature over a range of coupling, we have shown that the model exhibits a deconfinement transition when the 't Hooft coupling is of order one. We found that critical value of the coupling in the quenched case was of the same order as in the full theory. At first glance this might be surprising because naively extrapolating the weak coupling result to the temperature at which we did our simulations would suggest a difference of more than an order of magnitude. However, at sufficiently high temperatures we expect the dynamics to be dominated by bosonic zero modes on the thermal circle [31], in which case the critical coupling of the full theory should converge to that of the quenched theory. The fact that our results seem to indicate this convergence is setting in at  $\mu\beta = 1$  suggests that in future investigation of the phase diagram, it is sufficient to study the quenched theory if we are interested in inverse temperatures that sit below  $\beta\mu = 1$  in figure 8.

In principle, the thermal ensemble should sample states near each of the discrete

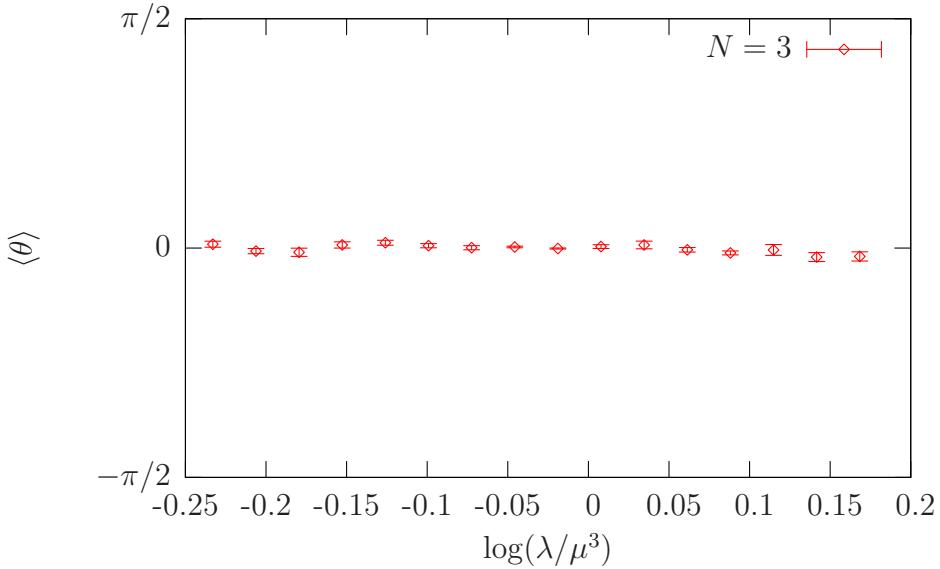


Figure 7: The phase of the Pfaffian for the full theory with gauge group  $SU(3)$ , fixed  $\mu\beta = 1$ , and five lattice points. The phase is clearly small in the region of parameter space we are exploring.

vacua. In the large  $N$  limit, since the number of vacua grows exponentially with  $N$ , we expect that the moduli space problem encountered in simulations of the BFSS model will reappear in the PWMM. The effect of the additional vacua can be enhanced or suppressed depending on the physical question of interest using umbrella sampling. Though we focussed on the deconfinement transition, and the Polyakov loop is degenerate for non-trivial vacua, we did not seem to need umbrella sampling to restrict to the trivial vacuum. A full study of the behaviour of the scalar fields is interesting, however, and we leave this for future work.

Though our simulation does not cover the entire  $\beta - \lambda$  plane, it is tempting to speculate that the phase diagram is of the form in figure 8. This seems most natural from the point of view of gauge/gravity duality. If the transition at strong coupling was not connected to that at weak coupling, for example, that would suggest the existence of some new gravity solution.

We have focussed on one particular aspect of the finite temperature strong coupling dynamics of the PWMM, however, there are many other questions of interest; we will mention two. In the limit of  $\mu \rightarrow 0$  at fixed  $\beta$  and  $\lambda$ , the PWMM becomes the BFSS matrix model. This model has a well-defined dual black hole solution, and it would be interesting to explore this limit to determine if the anticipated black hole thermodynamics is reproduced by the strongly coupled matrix model. Also of interest is to simulate the theory in the limit in which the PWMM becomes  $\mathcal{N} = 4$  SYM on  $R \times S^3$ ; results of non-lattice techniques in this direction were presented in [32].

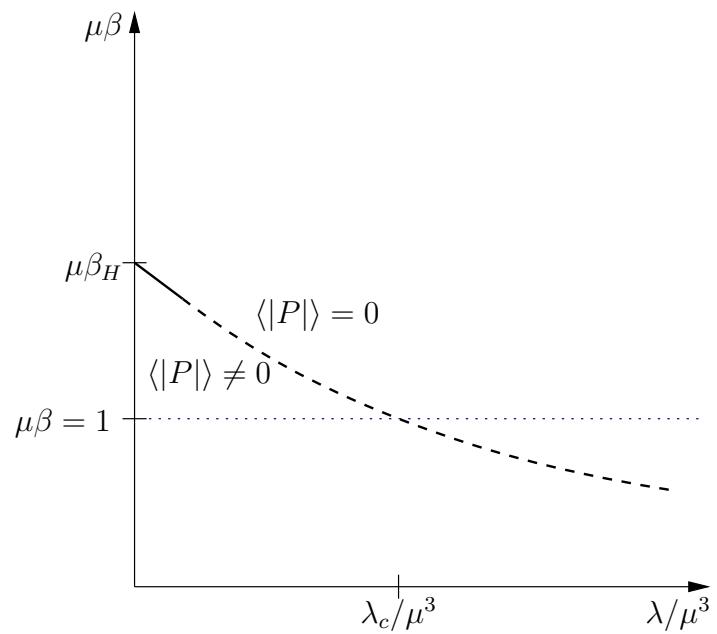


Figure 8: Hypothesized schematic picture of the phase diagram of the PWMM. Our simulation scans along the blue dotted line where we are able to identify a deconfinement transition. The heavy solid black line represents the known weak coupling result. The dashed black line represents a suggestion of what the phase diagram could look like interpolating between these points.

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